

Linear independence of nearest-neighbor valence-bond states on the kagome lattice and construction of $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonian with a Sutherland-Rokhsar-Kivelson quantum liquid ground state

Alexander Seidel

*Department of Physics and Center for Materials Innovation, Washington University, St. Louis, Missouri 63136, USA
and Kavli Institute for Theoretical Physics, Santa Barbara, California 93106, USA*

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A class of local $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonians is studied that has ground states within the space of nearest-neighbor valence-bond states on the kagome lattice. Cases include “generalized Klein” models without obvious non-valence-bond ground states, as well as a “resonating valence-bond” Hamiltonian whose unique ground states within the nearest-neighbor valence-bond space are four topologically degenerate “Sutherland-Rokhsar-Kivelson” (SRK) -type wave functions, which are expected to describe a gapped \mathbb{Z}_2 spin liquid. The proof of this uniqueness is intimately related to the linear independence of the nearest-neighbor valence-bond states on quite general and arbitrarily large kagome lattices, which is also established in this work. It is argued that the SRK ground states are also unique within the entire Hilbert space depending on properties of the generalized Klein models. Applications of the strategies developed in this work to other lattice types are also discussed.

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I. INTRODUCTION

A. General Motivation

For over 70 years, the study of magnetism has played a pivotal role in defining paradigms in condensed-matter physics. Heisenberg’s original notion of a local exchange interaction and its generalizations have given rise to a rich and interesting class of Hamiltonians, whose exploration has led to great success both in understanding known magnetic phenomena, as well as anticipating new ones. This approach has been successful to such a degree that sometimes even models studied out of purely academic interest have shed light on later observed experimental phenomena. Prominent examples include the one-dimensional (1D) Heisenberg chain with antiferromagnetic exchange constant, which was found to be exactly solvable by Bethe in the 30s.¹ This work did not only initiate the still thriving field of integrable model systems but founded the theory of 1D quantum antiferromagnets long before their experimental discovery in systems such as Sr_2CuO_3 .² Another example is the Shastry-Sutherland model,³ whose study predated the discovery of a closely related valence-bond solid ground state in the compound $\text{SrCu}_2(\text{BO}_3)_2$.^{4,5} Despite these successes, the study of generic Hamiltonians describing locally interacting spins on a lattice remains a highly challenging task. The low-energy properties of a given model are often difficult to extract with great confidence. However, the common scenario in dimensions greater than one is that the ground state of a system of lattice spins governed by a local Hamiltonian will display some form of long-range order, by either breaking $SU(2)$ -spin rotational symmetry or lattice translational symmetry, or both (see, e.g., Ref. 6). This phenomenon is well understood through the general framework that has been developed around Landau’s notion of spontaneous symmetry breaking.⁷ In recent decades, however, workers in the field have increasingly been interested in conditions that allow the ground state of a spin system to remain quantum disordered.

Anderson has argued that a quantum magnet may refrain from symmetry breaking even at $T=0$ as a result of quantum fluctuations and/or frustration.⁸ He later proposed that the resulting “resonating valence-bond (RVB) spin-liquid” state may be thought of as the ideal parent state for the cuprate superconductors.⁹ This proposal has led to considerable efforts in searching for such a state. Both on the theoretical and on the experimental side this search turned out to be a formidable challenge. Experimentally, there has been much recent excitement about compounds featuring two-dimensional (2D) layers of spin- $\frac{1}{2}$ degrees of freedom forming a triangular^{10,11} or kagome^{12–15} lattice, with no apparent sign of order at low temperatures. On the theoretical side, a standard way to establish the existence of a phase is to identify a low-energy effective-field theory describing the universal properties of the phase, together with special solvable points in the phase diagram of some microscopic Hamiltonian that can be demonstrated to display some of these universal features. This strategy has been very successful in a variety of contexts, such as interacting 1D quantum systems or the fractional quantum-Hall effect. While a thorough understanding of the phase is usually possible only through the field-theoretic description, its *existence* in a certain microscopic setting may be questionable until a microscopic realization is found, either in theory or in experiment. This is due to the fact that mappings between microscopic and field-theoretic descriptions, while extremely powerful, are necessarily non-rigorous. On the other hand, the construction of exactly solvable higher dimensional spin Hamiltonians is generally difficult. Recent successes along these lines with regard to the spin-liquid problem will be reviewed in Sec. I B. Currently, however, there is (to the best of my knowledge) no concrete example for an $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonian on a simple lattice, whose ground-state properties are analytically accessible and agree with those of a gapped (topological) spin liquid. The main purpose of this paper is to propose such a Hamiltonian.

B. Klein models and quantum dimer models

The difficulty of constructing solvable models of spin liquids motivated Kivelson and Rokhsar to consider the problem in a simplified Hilbert space.¹⁶ Their model features “dimer” degrees of freedom on the links of a lattice, which represent singlet bonds between nearest-neighbor sites. These singlets are the “valence bonds” of Anderson’s original proposal.⁸ Endowed with simple dynamics, the quantum dimer models display phase diagrams that may be interpreted in terms of singlet spin states with short-range spin-spin correlations. (For recent reviews, see Refs. 17 and 18). While the original quantum dimer model on the square lattice (as well as other bipartite lattices) does have an exactly solvable point with a liquid ground state, this turned out to be a critical-point separating “valence-bond solid” phases with broken translational symmetry.^{19–22} However, over a decade later, Moessner and Sondhi²³ found that a similar model on the triangular lattice has a solvable point corresponding to a stable quantum liquid phase. Subsequently, similar findings were also made for the kagome lattice.²⁴ While these findings are realizations of topologically ordered²⁵ quantum liquids in lattice models, their implications for quantum spin systems are not immediate. This is not only due to the truncation of the Hilbert space to nearest-neighbor valence-bond (NNVB) states but perhaps even more so due to the assumed orthogonality of different dimer coverings in quantum dimer models. In contrast, the associated valence-bond states are not orthogonal. In fact, even their linear independence is far from obvious. Though the linear independence of NNVB states has so far been proven only for the square and honeycomb lattice by Chayes *et al.*,²⁶ it is assumed to hold more generally, e.g., based on numerical studies.^{27,28} This matter is of fundamental interest in any attempt to formulate effective theories for frustrated spin- $\frac{1}{2}$ magnets within the NNVB subspace, except perhaps when a large- N point of view²⁹ is adopted. Furthermore, arguments assuming linear-independence properties of NNVB states have been employed in entropic considerations for frustrated spin- $\frac{1}{2}$ systems, e.g., on the kagome lattice,³⁰ as well as others.³¹ For the kagome case, the linear independence of the nearest-neighbor valence-bond states will be established in this work.

Although exact mappings between the dynamics of quantum dimer models and that of $SU(2)$ -invariant spin- $\frac{1}{2}$ quantum magnets on the same lattice do not exist, more general mappings have been applied successfully. In Ref. 32, a dimer model on the triangular lattice has been mapped onto a model of spin degrees of freedom in a highly anisotropic kagome antiferromagnet. A mapping that preserves $SU(2)$ invariance has been performed in Ref. 33, where highly decorated lattices are considered. It is interesting to note that for some higher spin systems, the problem of writing down $SU(2)$ -invariant Hamiltonians with unique spin-liquid ground states has been solved over 20 years ago through the well-known AKLT construction.³⁴ These states have been shown to be gapped in 1D,³⁵ and are believed to be gapped in higher dimensions as well. In contrast, through generalizations^{36,37} of the Lieb-Schultz-Mattis theorem,³⁸ it is nowadays well understood that spin- $\frac{1}{2}$ liquids must be ei-

ther gapless or have a nontrivial topological ground-state degeneracy on lattices with an odd number of sites per unit cell (cf. also Refs. 39–42). The latter implies that they are topologically ordered.²⁵ There is much interest in such topologically ordered phases recently motivated in part by their proposed use in quantum computation.⁴³ A significant number of solvable lattice models with topologically ordered ground states are known.^{43–47} Some of these models can be naturally cast in terms of spin- $\frac{1}{2}$ degrees of freedom but will lack $SU(2)$ invariance in this language. On the other hand, a parent Hamiltonian for a topologically ordered $SU(2)$ -invariant *chiral* spin-liquid state has recently been discussed,^{48,49} where time-reversal symmetry is explicitly broken.

An alternative route to the construction of $SU(2)$ -invariant models with spin-liquid ground states is based on a class of models introduced by Klein.⁵⁰ The Klein models have an extensive ground-state entropy with all NNVB states being ground states. When perturbed, one expects the ground state(s) to be a coherent superposition of NNVB states, to good approximation. This is particularly so if *all* ground states at the Klein point are of valence-bond type.²⁶ It is, however, nontrivial to determine the nature of such perturbed Klein models.

C. Outline

In this work, a local $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonian will be constructed on the kagome lattice, which has ground states that are spin- $\frac{1}{2}$ realizations of the dimer liquid at the solvable “Rokhsar-Kivelson” (RK) point of the quantum dimer model on the same lattice.²⁴ Some rigorous statements about the uniqueness of these ground states are deeply related to the linear independence of the NNVB states on general kagome-type lattices. As a byproduct, the linear independence of these states will be proven in the beginning of this paper. In Sec. II, this linear-independence property will be stated precisely and various definitions are introduced that will be useful in the remainder of the paper. The linear-independence property is then proven in Sec. III A. The proof is based on the observation that the linear-independence property for quite general and arbitrarily large kagome-type lattices can be reduced to a property of finite clusters. This property can be verified numerically, or alternatively, by analytic means. Both methods have been carried out, although the details of the analytic method (for which there is no real need except from a purist viewpoint) are not presented here to keep the length of the paper within certain bounds. Possible generalizations of the linear-independence theorem and application of the present method to different lattice types are then discussed in Sec. III B. Section IV is devoted to the construction of local $SU(2)$ -invariant Hamiltonians with valence-bond-type ground states. In Sec. IV A, Hamiltonians are constructed such that any NNVB state on the kagome lattice is a ground state, and, unlike for the Klein model on the same lattice, there are no *obvious* other ground states. In Secs. IV B 1–IV B 3, this construction is further generalized to allow no obvious ground states other than the Sutherland-Rokhsar-Kivelson (SRK) wave functions which are akin to similar states at the exactly solvable point of the

kagome lattice quantum dimer model.²⁴ In Sec. IV B 4 it is then proven that these are in fact the only ground states within the manifold of NNVB states. In Sec. IV B 5, a strategy will be discussed to prove that the SRK wave functions are the only ground states even within the full Hilbert space. This question will be related to properties of the generalized Klein models of Sec. IV A, which are not proven here. In Sec. V it will be argued that the model constructed in Sec. IV B likely describes a \mathbb{Z}_2 topological liquid. Section VI concludes the paper. A technical detail regarding the uniqueness of the SRK ground states is presented in the Appendix.

I finally remark upon the relation of the present work to that of Ref. 51, where spin Hamiltonians whose ground states are valence-bond realizations of RK dimer states have also been constructed. In Ref. 51, the corresponding dimer states describe critical points in the phase diagram of their dimer Hamiltonians. In contrast, the dimer wave function that inspired the ground states of the model constructed here corresponds to a stable \mathbb{Z}_2 topological liquid. In Sec. V, it will be argued that the properties of the dimer liquid will largely carry over to the spin-wave function. Similar arguments have been made for the critical cases in Ref. 51 and have been put forth early on by Sutherland⁵² for the square-lattice case. The strategy developed here is applicable to RK points on different lattices as well, including the square and honeycomb cases of Ref. 51, although the present construction would not in any obvious way give rise to the same Hamiltonians discussed there. Application of the present construction to other lattices will be further discussed in Sec. IV B 5.

II. DEFINITIONS AND STATEMENT OF THE LINEAR-INDEPENDENCE PROPERTY

I will start by defining some terms that will be useful in the following. In general, I will use the term *lattice* to refer to any collection of discrete points or lattice sites, \mathcal{L} , which have a topology imposed through the notion of *nearest neighbors*. The set of nearest neighbors of a lattice site i is called its *neighborhood* and is denoted by $\mathcal{N}(i)$. If j is a nearest neighbor (NN) of i , then i is a NN of j , and the (unordered) pair (i, j) is called a link of the lattice. I will say that i and j are the sites “touched” by the link (i, j) . While many aspects of this work are expected to generalize to other lattices of interest, I will mainly focus on kagome-type lattices (Fig. 1). For now, the lattice \mathcal{L} may be any finite subset of the infinite 2D kagome lattice, or, more generally, of an arbitrarily large but finite kagome lattice with toroidal periodicity. More general global topologies are also possible as long as the local structure is that of a kagome lattice, as will become clear in the following. For greater simplicity, however, I will defer the discussion of such cases to Sec. III B.

A *cell* may be any subset \mathcal{C} of a given lattice \mathcal{L} , but will usually refer to reasonably small and well-connected units such as shown in Fig. 2, with nonvanishing interior (to be defined next). For kagome-type lattices, I define the *interior* \mathcal{C}° of the cell \mathcal{C} as the set of all points in \mathcal{C} that have four nearest neighbors also contained in \mathcal{C} . Likewise, the *boundary* $\partial\mathcal{C}$ of \mathcal{C} consists of those points in \mathcal{C} that are not interior.

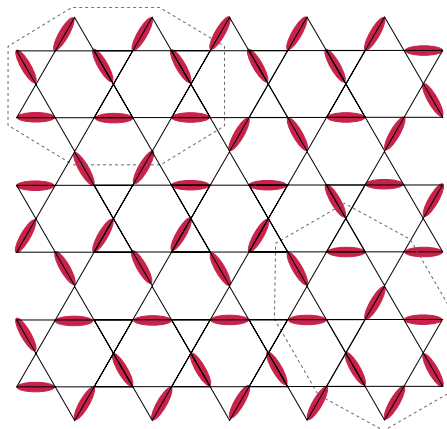


FIG. 1. (Color online) A kagome lattice. Some 19-site cells with the topology of Fig. 2(b) are indicated. The lattice is “regular” as defined in the text. It is also fully dimerizable, as indicated by the shown dimer covering.

A *dimer covering* D of the cell \mathcal{C} is a set of disjoint (!) links between sites in \mathcal{C} such that each interior site of \mathcal{C} is touched by one link in D . I will also refer to the links of D as the *dimers* of the covering. The support of the dimer covering D , denoted $\text{supp}(D)$, is the set of all sites belonging to a dimer in D . Note that $\mathcal{C}^\circ \subset \text{supp}(D)$ by definition, but the opposite inclusion need not hold, since $\text{supp}(D)$ will in general contain some boundary points of \mathcal{C} as well, cf. Fig. 2. Likewise, I introduce the set of *free sites* $\text{free}(D, \mathcal{C}) = \mathcal{C} \setminus \text{supp}(D)$ of the dimer covering D , which contains all (boundary) sites in \mathcal{C} that are not touched upon by the covering. The dependence on \mathcal{C} may be suppressed when it is clear what cell is referred to, and I then denote the free sites of the covering simply by $\text{free}(D)$. By $\mathcal{D}(\mathcal{C})$ I denote the set of all dimer coverings of \mathcal{C} , and by $\mathcal{D} \equiv \mathcal{D}(\mathcal{L})$ the set of all dimer coverings of the lattice. For every subset \mathcal{S} of \mathcal{L} , $\mathcal{H}(\mathcal{S})$ denotes the Hilbert space

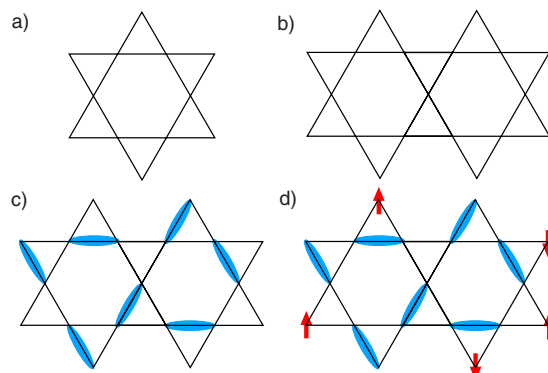


FIG. 2. (Color online) [(a) and (b)] Some cells of the kagome lattice. The cell shown in (b) is the smallest cell for which the linear-independence property stated in the text holds. It is thus the smallest regular kagome lattice as defined in the text. (c) A dimer covering D of the cell shown in (b). Note that according to the definition used here, boundary sites need not participate. (d) A state compatible with the dimer covering D shown in (c), i.e., an element of $\mathcal{H}(|D\rangle)$. Dimers correspond to singlets formed by spin- $\frac{1}{2}$ degrees of freedom on neighboring sites, whereas boundary sites not touched by dimers are allowed to be in an arbitrary state.

obtained by associating a spin- $\frac{1}{2}$ degree of freedom with every site in \mathcal{S} , and $\mathcal{H} \equiv \mathcal{H}(\mathcal{L})$ the Hilbert space associated with the entire lattice. Since $\mathcal{H}(\mathcal{S}) \cong \mathbb{C}^{2^n}$, in general, where n is the number of sites in \mathcal{S} , it is natural to take $\mathcal{H}(\mathcal{S}) \equiv \mathbb{C}$ should \mathcal{S} by some chance be empty. With this convention, we can always write

$$\mathcal{H}(\mathcal{C}) = \mathcal{H}[\text{supp}(D)] \otimes \mathcal{H}[\text{free}(D)] \quad (1)$$

when D is a dimer covering of \mathcal{C} . More generally, consider a cell \mathcal{C} that is the disjoint union of two subcells \mathcal{C}' and \mathcal{C}'' , such that $\mathcal{H}(\mathcal{C}) = \mathcal{H}(\mathcal{C}') \otimes \mathcal{H}(\mathcal{C}'')$. Given a (pure) state $|s'\rangle \in \mathcal{H}(\mathcal{C}')$, we can define a subspace $\mathcal{H}(|s'\rangle, \mathcal{C})$ of $\mathcal{H}(\mathcal{C})$ consisting of all states that are *compatible* with the state $|s'\rangle$,

$$\mathcal{H}(|s'\rangle, \mathcal{C}) = \text{span}(|s'\rangle) \otimes \mathcal{H}(\mathcal{C}'') = \{|s'\rangle \otimes |s''\rangle : |s''\rangle \in \mathcal{H}(\mathcal{C}'')\}, \quad (2)$$

where $\text{span}(\cdot)$ denotes the linear span of the vector(s) enclosed by the brackets. Again, if it is clear what cell is referred to, we may write $\mathcal{H}(|s\rangle)$ instead of $\mathcal{H}(|s\rangle, \mathcal{C})$. See Fig. 2 for an example.

Let us fix a cell \mathcal{C} for the moment. For a given dimer covering D of \mathcal{C} we denote by $|D\rangle$ a state in $\mathcal{H}[\text{supp}(D)]$ in which any two spins belonging to the same link in D form a singlet or valence bond (VB). The state $|D\rangle$ is just a realization of the dimer covering D through valence bonds. Then $\mathcal{H}(|D\rangle)$, as defined in Eq. (2), consists of all states that are compatible with the dimer covering D . A general state belonging to $\mathcal{H}(|D\rangle)$ for some dimer covering D will have every internal site participating in a nearest-neighbor singlet, while boundary sites left “free” by the covering D may be in any arbitrary state [cf. Fig. 2(d), but note that the free sites could in general be entangled]. The space spanned by all states of this kind will be called the space of *valence-bond states* of the cell \mathcal{C} , denoted by $\text{VB}(\mathcal{C})$,

$$\text{VB}(\mathcal{C}) = \sum_{D \in \mathcal{D}(\mathcal{C})} \mathcal{H}(|D\rangle), \quad (3)$$

where the sum denotes the linear span of the spaces summed over. Note that the space $\text{VB}(\mathcal{C})$ is $SU(2)$ invariant. This is so since for $|s'\rangle = |D\rangle$, both factors in the first line of Eq. (2) are $SU(2)$ invariant and hence $\mathcal{H}(|D\rangle)$ is $SU(2)$ invariant for each D .

It is clear from the definition how to write down a set of states that linearly generate the space $\text{VB}(\mathcal{C})$. For each dimer covering D , we denote by $|\psi_{D,j}\rangle$ a basis of the space $\mathcal{H}[\text{free}(D)]$ of spins not touched by the covering. Here, j runs from 1 to $n_D \equiv 2^{|\text{free}(D)|}$. By definition, then, the following set of states,

$$\mathcal{B}(\mathcal{C}) = \{|D\rangle \otimes |\psi_{D,j}\rangle : D \in \mathcal{D}(\mathcal{C}), j = 1, \dots, n_D\}, \quad (4)$$

linearly generates the space $\text{VB}(\mathcal{C})$ of valence-bond states of the cell \mathcal{C} . We will be interested in the question for what cells \mathcal{C} all states in $\mathcal{B}(\mathcal{C})$ are linearly independent. Note that if \mathcal{C} is the entire lattice \mathcal{L} , and \mathcal{L} has no boundary, then there are no free sites for any D , and the valence-bond states $\mathcal{B}(\mathcal{L})$ consist of all NN valence-bond “dimerizations” of the lattice in the usual sense. In the presence of a nontrivial boundary, however, states are also admitted where some boundary sites do

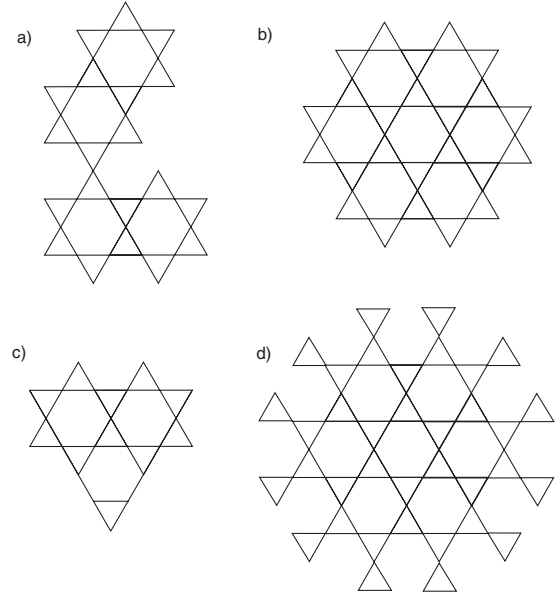


FIG. 3. Regular and nonregular kagome lattices. [(a) and (b)] Regular kagome lattices. (b) is fully dimerizable, (a) is not, due to an odd number of sites. [(c) and (d)] Nonregular kagome lattices.

not participate in the NN singlet bonding. Since more states are being included, the statement of their linear independence is a stronger one than if only “true” dimerizations were considered. Indeed, the great advantage of including these states is that the linear independence of the set $\mathcal{B}(\mathcal{C})$ for certain small cells \mathcal{C} is now so strong a statement that it immediately carries over to entire lattice, as will be demonstrated shortly below. This fact, and the observation that some fairly small cells already have this linear-independence property, constitute the main ingredients of the proof constructed in the following section. For simplicity, I will now focus on lattices \mathcal{L} that are subsets of some finite 2D kagome lattice with doubly periodic (toroidal) boundary conditions. Any such lattice will be called a *regular kagome lattice*, if and only if any link (i,j) of \mathcal{L} belongs to a 19-site cell \mathcal{C} contained in \mathcal{L} that has the topology shown in Fig. 2(b). In general, the cell \mathcal{C} will of course depend on the link (i,j) and need not be unique. Examples for regular kagome lattices include such interesting cases as the lattice depicted in Fig. 1, those in Figs. 3(a) and 3(b), as well as any sufficiently large kagome lattice with toroidal periodic boundary conditions. The following theorem is the main result on linear independence of valence-bond states, and will be proven in Sec. III:

Theorem I. For any regular kagome lattice \mathcal{L} , the set of valence-bond states $\mathcal{B}(\mathcal{L})$ is linearly independent. Furthermore, one is sometimes interested only in *full dimerizations* of the lattice, where every site is touched by a dimer, including all boundary sites. I will call the lattice *fully dimerizable* if and only if there is a way to group all sites into disjoint pairs of nearest neighbors. Any such way will be called a full dimerization. Full dimerizations of regular lattices are special cases of dimer coverings as defined above. The corresponding valence-bond states are thus a subset of the set $\mathcal{B}(\mathcal{L})$, and one obtains the following theorem as simple corollary of Theorem I.

Theorem II. For any fully dimerizable regular kagome lattice, the valence-bond states associated with full dimerizations are linearly independent.

III. PROOF OF THE LINEAR-INDEPENDENCE PROPERTY

A. Regular kagome lattices

One immediate consequence of Theorem I is that the 19-site cell shown in Fig. 2(b) does by itself already have the stated linear-independence property. In this section, \mathcal{C} will always refer to a cell of this topology. Indeed, this seems to be the smallest cell of the kagome lattices for which the set $\mathcal{B}(\mathcal{C})$, Eq. (4), is linearly independent [see the appendix for “spoilers” when \mathcal{C} is chosen to be the cell in Fig. 2(a)]. Conversely, it will be shown in the following that once this property is established for the 19-site cell, it immediately generalizes to any kagome lattice that is regular in the sense defined above. Although I will focus on the kagome case, the argument for this is very general and should allow for rather direct generalizations to other cases of interest: Once the linear independence of $\mathcal{B}(\mathcal{C})$ is established for suitable “building blocks” \mathcal{C} of the lattice, it will generalize to the lattice as a whole. I will call these building blocks the “bricks of linear independence” of the lattice \mathcal{L} . These bricks may quite generally be reasonably small, as is at least suggested by the present example: The set $\mathcal{B}(\mathcal{C})$ for the 19-site cell in Fig. 2(b) consists of 13 120 states, whose linear independence can be verified numerically or using computer algebra in a straightforward manner. This task amounts to checking that a suitably defined integer overlap matrix has full rank. I have used the LINBOX package⁵³ for this purpose. However, an analytic proof of the linear independence of $\mathcal{B}(\mathcal{C})$ is also possible⁵⁴ using the results stated in the appendix for the 12-site cell of Fig. 2(a). The latter can be derived using Rumer-Pauling valence-bond diagrams.^{55–58} It is of little importance, however, whether the linear-independence property of the set $\mathcal{B}(\mathcal{C})$ is obtained by analytic or numerical means. Once this property is established, the same property follows for arbitrarily large regular lattices, as will be shown in the following.

According to the above, the set $\mathcal{B}(\mathcal{C})$ defines a basis for the space of valence-bond states $\text{VB}(\mathcal{C})$, Eq. (3). If we label these basis states by an index b and write $|b\rangle$ for the elements of $\mathcal{B}(\mathcal{C})$, we can therefore define a set of linear projection operators P_b with the following properties,

$$P_b|b'\rangle = \delta_{b,b'}|b\rangle, \quad P_b P_{b'} = \delta_{b,b'} P_b. \quad (5)$$

The action of the operators P_b within the space $\text{VB}(\mathcal{C})$ is fully defined by Eq. (5).⁵⁹ The existence of such operators is guaranteed by the linear independence of the set $\mathcal{B}(\mathcal{C})$. Note that the P_b are not Hermitian since the states $|b\rangle$ are not orthogonal.

As a next step, I define projection operators that leave all valence-bond states corresponding to a given dimer covering D invariant and annihilate all valence-bond states corresponding to dimer coverings $D' \neq D$. This is easily accomplished in terms of the operators defined by Eq. (5). Recall

that each of the states $|b\rangle \in \mathcal{B}(\mathcal{C})$ is of the form $|b\rangle = |D\rangle \otimes |\psi_{D,j}\rangle$, where $|\psi_{D,j}\rangle$ denotes the state of the sites not touched by the dimer covering. We can thus write $P_b \equiv P_{D,j}$. The operators

$$P_D = \sum_j P_{D,j} \quad (6)$$

then satisfy

$$P_D|D'\rangle \otimes |\psi_{D',j}\rangle = \delta_{D,D'}|D'\rangle \otimes |\psi_{D',j}\rangle, \quad \text{for any } j,$$

$$P_D P_{D'} = \delta_{D,D'} P_D. \quad (7)$$

The idea is now to observe that the successive action of operators of this kind defined on various 19-site bricks of some larger regular lattice \mathcal{L} can “single out” any given dimer covering of this lattice. Detailed arguments are given in the following.

Consider now a regular kagome lattice \mathcal{L} . By definition, every link of the regular lattice belongs to at least one 19-site brick of the topology shown in Fig. 2(b). For definiteness, one may consider the brick shown in the upper left-hand corner of the lattice in Fig. 1. Let now $D_C, D'_C \in \mathcal{D}(\mathcal{C})$ be dimer coverings of the brick \mathcal{C} . Further consider any state on \mathcal{C} that is compatible with the dimer covering D_C in the sense that it factorizes into the valence-bond state $|D_C\rangle$ on $\text{supp}(D_C)$ and any other state $|S\rangle$ on $\mathcal{C} \setminus \text{supp}(D_C)$, i.e., a state of the form $|D_C\rangle \otimes |S\rangle$. The properties (7) of the operators $P_{D'_C}$, which act on the brick \mathcal{C} , immediately imply

$$P_{D'_C}|D_C\rangle \otimes |S\rangle = \delta_{D_C,D'_C}|D_C\rangle \otimes |S\rangle. \quad (8)$$

To see this, all we need to do is to expand the state $|S\rangle$ in a basis of the form $|\psi_{D_C,j}\rangle \otimes |S'_j\rangle$, where the first factor is a basis state of $\mathcal{H}[\text{free}(D_C, \mathcal{C})]$ as it appears in Eq. (7), and the second factor is an element of some basis of $\mathcal{H}(\mathcal{C})$. Equation (8) then immediately follows from Eq. (7).

Next consider a particular dimer covering D of \mathcal{L} , and a valence-bond state compatible with D , $|D\rangle \otimes |\psi_{D,j}\rangle$. Again, $|\psi_{D,j}\rangle$ is a state of $\mathcal{H}[\text{free}(D, \mathcal{L})]$, chosen from some arbitrary basis. The dimer covering D induces a dimer covering of \mathcal{C} , the *restriction* D_C of D on \mathcal{C} , consisting of all dimers of D that are fully contained in \mathcal{C} (cf., e.g., Fig. 1). Then, the valence-bond state $|D\rangle \otimes |\psi_{D,j}\rangle$ is of the form $|D_C\rangle \otimes |S\rangle$ displayed in Eq. (8). To make this explicit, one may introduce the *complement* of the dimer covering D_C in D , which consists of those dimers in D that are not contained in D_C , and denote it by $\overline{D_C}$. Then by definition

$$|D\rangle = |D_C\rangle \otimes |\overline{D_C}\rangle. \quad (9)$$

Hence, the state $|D\rangle \otimes |\psi_{D,j}\rangle$ is of the stated form with $|S\rangle = |\overline{D_C}\rangle \otimes |\psi_{D,j}\rangle$, and from Eq. (8) we have

$$P_{D'_C}|D\rangle \otimes |\psi_{D,j}\rangle = \delta_{D_C,D'_C}|D\rangle \otimes |\psi_{D,j}\rangle \quad (10)$$

for any dimer covering D'_C of the brick \mathcal{C} , where again the operator $P_{D'_C}$ acts on this particular brick. In the following, the case of interest will be that D'_C is also obtained as the restriction on \mathcal{C} of some dimer covering D' of \mathcal{L} . In words, Eq. (10) then says that the valence-bond state $|D\rangle \otimes |\psi_{D,j}\rangle$

will survive the action of $P_{D'_C}$ unaltered if the dimer coverings D and D' locally look the same within the brick \mathcal{C} and will be annihilated if not. Here, “looking the same locally” means that all dimers fully contained in \mathcal{C} are identical.

The proof of the linear independence of the set $\mathcal{B}(\mathcal{L})$ is now trivial. Suppose we have a linear combination of states in $\mathcal{B}(\mathcal{L})$ that vanishes identically

$$\sum_{D \in \mathcal{D}(\mathcal{L})} \sum_{j=1}^{n_D} \lambda_{D,j} |D\rangle \otimes |\psi_{D,j}\rangle = 0. \quad (11)$$

We need to show that this implies that all $\lambda_{D,j}$ are zero. First we successively act on Eq. (11) with all operators of the form $P_{D'_C}$ for a fixed dimer covering D' , and for all bricks \mathcal{C} of the lattice, i.e., we act on Eq. (11) with

$$\prod_{\mathcal{C}} P_{D'_C}. \quad (12)$$

This will eliminate all D with $D \neq D'$ from Eq. (11). For, if $D \neq D'$, there must be a link of the lattice which, say, belongs to D but not to D' . Since the lattice is regular, there is a brick \mathcal{C} containing this link. The operator $P_{D'_C}$ corresponding to this brick will then annihilate all states of the form $|D\rangle \otimes |\psi_{D,j}\rangle$. Note that the operators $P_{D'_C}$ commute within $\text{VB}(\mathcal{L})$, since by Eq. (10), $\text{VB}(\mathcal{L})$ is the span of a common set of eigenstates. It follows, then, that the action of Eq. (12) on Eq. (11) annihilates all states with $D \neq D'$, whereas all states of the form $|D'\rangle \otimes |\psi_{D',j}\rangle$ are invariant under this action. We are thus left with an equation of the form

$$\sum_{j=1}^{n_{D'}} \lambda_{D',j} |D'\rangle \otimes |\psi_{D',j}\rangle = 0. \quad (13)$$

But since the states appearing in here only differ through the states $|\psi_{D',j}\rangle$ on $\text{free}(D')$, and the states $|\psi_{D',j}\rangle$ are linearly independent by definition, it follows that $\lambda_{D',j} = 0$ for any j . Since D' was arbitrary, all coefficients in Eq. (11) must vanish identically. This completes the proof of Theorem I.

B. Further generalizations

For reasons of simplicity, Theorems I and II have not been stated in the most general form one could imagine. For one, the restriction to sublattices of finite periodic kagome lattices, which have toroidal topology, is not strictly necessary. This was done because such lattices do not require a very technical definition and are certainly general enough for most purposes. Quite similarly, when talking about two-dimensional manifolds one will think of these as being embedded into three-dimensional Euclidean space in simple enough cases, though this is not possible in general. Likewise, if one wants to apply the present results to a kagome-type lattice, say, of Möbius-strip topology, one would desire a more “local” definition of a regular kagome lattice. There is, in fact, no real obstacle in applying the present results to such a lattice since the operators $P_{D'_C}$ are entirely local. The only real requirement that must be made is thus that the lattice is “built up” from the 19-site bricks defined above.

One must define carefully, however, what built up means with regard to the neighborhood structure. For example, it is clear that one cannot apply the present results to three-dimensional stackings of planar kagome lattices without further ado, if there are links along the third direction. A sufficiently strong requirement would be that the lattice can be covered by bricks, such that for each lattice site, there is a brick containing the entire neighborhood of this site. More generally, it is only necessary that the restrictions of any dimer covering to all the bricks of the lattice uniquely determine the covering. For kagome-type lattices, using the arrow representation⁶⁰ (see Sec. IV B 4) one can show that this is already the case when each *site* of the lattice, rather than each link, is contained in a brick. This will be of some importance in Sec. IV B 4. (Cf. Fig. 9 and caption.) However, instead of trying to state Theorems I and II for the most general class of lattices, it may be more efficient to consider “nonregular” lattices on a case by case basis, and determine if the construction of the preceding section can be used to derive a linear-independence property. For example, the lattice in Fig. 3(d) is certainly not regular but Theorem II still applies. That is so because every full dimerization of this lattice necessarily has dimers on any of the links between two boundary sites. These dimers are thus mere spectators and have no bearing on the question of linear independence. The remaining lattice is, however, regular, and so the linear independence of the valence-bond states corresponding to full dimerizations still holds.

As stated, the present strategy to prove the linear independence can be readily applied to other lattices,⁶¹ provided that elementary “bricks” of the lattice can be identified which cover the lattice in the sense discussed above, and for which the linear independence of the set \mathcal{B} holds. It is in general possible to have more than one type of brick, which may be an advantage if the lattice is somewhat irregular. Obvious candidates to apply this method to include the triangular and the “pentagonal”³³ lattice, and also the square and the honeycomb lattice in the presence of periodic boundary conditions, which have so far been studied for open boundary conditions only.²⁶ One may be hopeful that the present strategy works at least for lattices with comparable or lower coordination number compared to the present case. While the coordination number of the triangular lattice is notoriously high, there is no *a priori* reason to exclude such cases from consideration. The same is true for some higher-dimensional lattices. A detailed case by case analysis is left for future studies.

IV. A HAMILTONIAN WITH SUTHERLAND-ROKHSAR-KIVELSON-RVB GROUND STATES

A. Generalized Klein Models

The final goal of this section is to construct a Hamiltonian whose ground states are special superpositions of NN valence-bond states. One natural starting point for this endeavor is the construction of a Hamiltonian whose ground-state sector contains the entire manifold of NN valence-bond states, but—if possible—no other states.

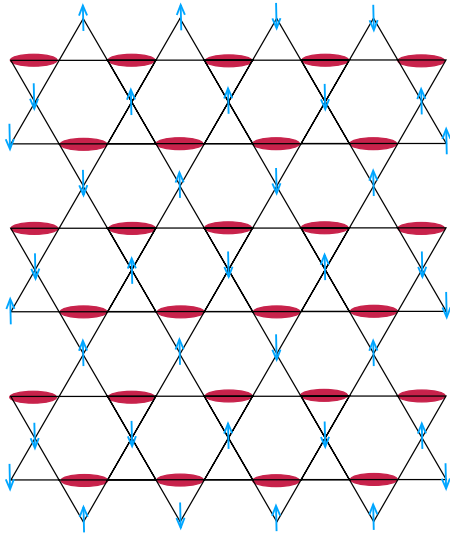


FIG. 4. (Color online) A ground state of the Klein model, Eq. (14), on the kagome lattice. The state has only one valence bond per unit cell and each bow tie of the lattice fully contains one such bond. The state obviously lies outside the space of valence-bond states since there are many sites not participating in valence bonds.

In a very influential work,⁵⁰ Klein discussed a strategy to construct models of this kind on general lattices. For each site i of the lattice, a projection operator P_i is considered which acts on a cell consisting of the site i and all its nearest neighbors, and projects onto the subspace of this cell that has maximum total spin. The Klein Hamiltonian,

$$H_{\text{Klein}} = \sum_i P_i, \quad (14)$$

then has a ground-state sector which includes all NN valence-bond states of the lattice. In some cases of interest, including the square lattice and the honeycomb lattice, it is believed that these valence-bond states are a complete set of ground states of Eq. (14), and a rigorous proof exists for the honeycomb case.²⁶ In other cases, however, there are obvious ground states outside the valence-bond subspace. The kagome lattice is an example of the latter kind, as demonstrated by the state shown in Fig. 4. One may ask whether it is possible to construct a local Hamiltonian for the kagome lattice such that the NNVB states span the entire ground-state sector. Since the Klein Hamiltonian (14) does not have this property, one will need to consider projection operators acting on cells larger than the five-site “bow ties” formed by a site i and its nearest neighbors. Possible examples include the cells shown in Fig. 5 and those in Figs. 2(a) and 2(b). For larger cells, with more than one internal site, it is not sufficient to project onto the maximum spin sector. One solution might be to project onto the sector with total spin $>N_s/2 - N_b$, where N_s is the number of sites of the cell and N_b the minimum number of valence bonds that must be entirely contained inside the cell in an arbitrary valence-bond states. Similar constructions, with different motivations in mind, have been explored in Refs. 31, 33, 62, and 63. However, since internal sites may or may not bond with each other, the

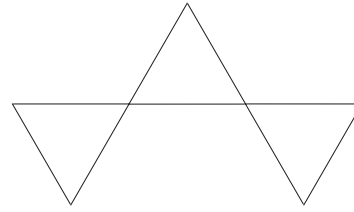


FIG. 5. A seven-site cell. The generalized Klein Hamiltonian (15) for this cell no longer has the ground state depicted in Fig. 4. Some non-valence-bond ground states do however remain.

number of valence bonds entirely contained within a cell may be larger for some valence-bond states than for others. In the present context, there is thus a more natural, and far more restrictive (for larger cells) way to construct local projection operators for which valence-bond states are (say) maximum eigenvalue eigenstates. For any cell \mathcal{C} consider the orthogonal projection operator $P_{\mathcal{C}}$ onto the subspace $\text{VB}(\mathcal{C})$, Eq. (3), and the Hamiltonian

$$H = - \sum_{\mathcal{C}} P_{\mathcal{C}}. \quad (15)$$

Here, the sum goes over all cells of a kagome lattice \mathcal{L} that have a certain topology, e.g., Fig. 2(a) and 5. Since $\text{VB}(\mathcal{C})$ is $SU(2)$ invariant so is each projection operator $P_{\mathcal{C}}$ and thus Hamiltonian (15). Furthermore, for any fixed cell \mathcal{C} , any valence-bond state will be a ground state of the operator $-P_{\mathcal{C}}$. This is so since any state in $\mathcal{B}(\mathcal{L})$ is of the form $|D_{\mathcal{C}}\rangle \otimes |S\rangle$, with $D_{\mathcal{C}}$ a dimer covering of \mathcal{C} and $|S\rangle$ a state on $\mathcal{L} \setminus \text{supp}(D_{\mathcal{C}})$, as explained above Eq. (9). It thus satisfies

$$P_{\mathcal{C}}|D_{\mathcal{C}}\rangle \otimes |S\rangle = |D_{\mathcal{C}}\rangle \otimes |S\rangle, \quad (16)$$

for reasons entirely analogous to those stated below Eq. (8). Valence-bond states are thus ground states of Eq. (15). If the cells on which the operators $P_{\mathcal{C}}$ are defined are just the five-site bow ties, Eq. (15) differs from the original Klein Hamiltonian only by a constant. For a sufficiently large choice of cell, however, there will be no obvious non-valence-bond-type ground states. This motivates the consideration of Eq. (15) for more general cells. These cells should have the property that they cover the considered lattice \mathcal{L} entirely. Moreover, it is possible to extend the sum in Eq. (15) over more than one type of cell. Furthermore, recall that the definition of valence-bond states in Sec. II admits states with dangling spins at the boundary $\partial\mathcal{L}$ of \mathcal{L} , as long as $\partial\mathcal{L}$ is not empty. However, a modification of the boundary terms in Eq. (15) may restrict the ground-state sector of Eq. (15) to valence-bond states corresponding to full dimerizations of the lattice. Specifically, for cells \mathcal{C} overlapping the lattice boundary $\partial\mathcal{L}$, one may restrict the dimer coverings D of \mathcal{C} defining the space $\text{VB}(\mathcal{C})$, Eq. (3), to those where every boundary site of \mathcal{L} is touched by a dimer. With this modification, only valence-bond states corresponding to full dimerizations will be obvious ground states of Eq. (15), even if the boundary is not empty. For a sufficiently large size of the cell \mathcal{C} , it seems likely that the valence-bond states and their linear combinations will be the unique ground states of Eq. (15). Note that the linear independence, or lack thereof, of the set $\mathcal{B}(\mathcal{C})$ is

irrelevant for Hamiltonian (15) to be meaningful, and is not in any obvious way related to the question of the uniqueness of the valence-bond ground states. For this reason we need not limit our attention to cells that contain the 19-site bricks studied in the preceding sections but smaller cells may suffice. For example, when the seven-site cell depicted in Fig. 5 is used to define Hamiltonian (15), some non-valence-bond ground states of the original Klein model will cease to be ground states of Eq. (15) (e.g., the state in Fig. 4), though not all. The smallest candidate for the cell C on which to base Hamiltonian (15) such that there are no non-valence-bond ground states is the 12-site star-shaped cell depicted in Fig. 2(a). A detailed study of the uniqueness of the valence-bond ground states for this choice of cell is left for future work. In the following, the ideas discussed in this section will be further generalized to allow the construction of a Hamiltonian whose only ground states, at least within the NN valence-bond basis, are the SRK-type RVB wave functions.

B. The RVB Hamiltonian

While the generalized Klein models constructed in the preceding subsection will have a lower ground-state degeneracy than the original Klein model (14), this degeneracy is still extensive. At the very least, every valence-bond state corresponding to an arbitrary dimer covering D of the lattice will be a ground state of any generalized Klein model. In this section, a local Hamiltonian is constructed with RVB-type ground states, which are certain superpositions of valence-bond states. One desires these RVB ground states to be unique and to describe quantum spin liquids with no spontaneously broken symmetry. A proof of the uniqueness within the restricted subspace of valence-bond states will be given in the following, whereas the generalization of the proof to the full Hilbert space will rely on properties of the generalized Klein models. In order to have a fair amount of confidence that the ground states describe spin liquids, the Hamiltonian will be designed such that its ground-state wave functions are spin- $\frac{1}{2}$ realizations of the Rokhsar-Kivelson point of the quantum dimer model of the on the kagome lattice.²⁴ These wave functions are in some sense the “prototypical” RVB-spin-liquid states, and at least for the quantum dimer model describe a \mathbb{Z}_2 quantum liquid.²⁴ Whether or not this is still the case when dimers are replaced by singlet valence bonds is a nontrivial question. It seems, however, likely that the answer is affirmative. The main technical difficulty in answering this question is the mismatch of the scalar product for corresponding states in the quantum dimer and the spin- $\frac{1}{2}$ Hilbert space. Further discussion of this issue will be given in Sec. V.

1. The quantum dimer model on the kagome lattice

In a seminal paper, Moessner and Sondhi²³ showed that the Rokhsar-Kivelson point of the quantum dimer model (QDM) on the triangular lattice has ground states describing a \mathbb{Z}_2 topological quantum liquid. A subsequent work by Misguich *et al.*²⁴ generalized these findings to the kagome lattice, which was found to have several additional attractive features. In the present context, the most important distinc-

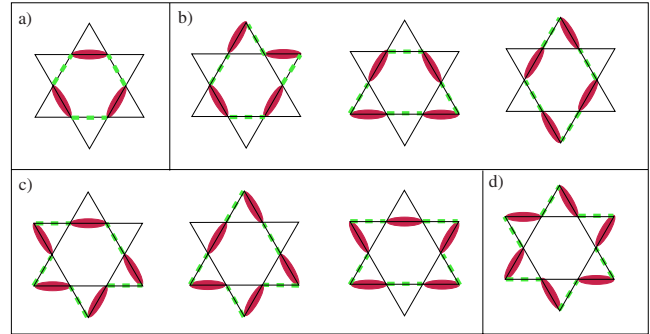


FIG. 6. (Color online) Eight different types of dimer loops around a central hexagon (Ref. 64). Loops may be formed by (a) three, (b) four, (c) five, or (d) six dimers. Each loop can be realized by two different dimer configurations related by a resonance move. Dashed lines indicate dimer configurations after a resonance move. The loops should be regarded as the transition graphs between the original and the resonated configuration, i.e., the set of all links carrying either a dimer or a dashed line. Using rotational symmetry, there are 32 different loops corresponding to 64 dimer configurations.

tive feature of the QDM on the kagome lattice is the fact that its RK point lies in the interior of the \mathbb{Z}_2 -liquid phase. This is in contrast to the triangular case, where the RK point of the QDM lies at a (apparently first-order) phase boundary. This will guarantee the uniqueness of the four “liquid” ground states of the model constructed here within the valence-bond subspace, and probably beyond, as argued in Sec. IV B 5.

The Hamiltonian of the QDM on the kagome lattice is a sum of operators acting on star-shaped 12-site cells as depicted in Fig. 2(a). In the following, the term “12-site cell” will always refer to cells of this topology. Any dimer covering of this cell, as defined in Sec. II, defines a loop⁶⁴ around the central hexagon, given by the (shortest) line connecting all points touched by a dimer, see Fig. 6. Each loop, on the other hand, is associated with two possible dimer coverings of the 12-site cell. These two dimer coverings are related by a shift of all dimers along the loop, exchanging links with and without a dimer. Such a shift will be referred to as a “resonance move” along the loop formed by the dimers. For definiteness, unless otherwise noted I will now assume a translationally invariant kagome lattice with toroidal topology. Every dimer covering D of the lattice gives rise to a covering D_C of every 12-site cell C and a corresponding loop. The Hamiltonian of the QDM of this lattice is a sum of operators acting on any 12-site cell of the lattice, where each operator performs a resonance move on the loop of dimers present on the cell it acts on. (See Ref. 24 for details.) The ground states of this Hamiltonian consist of equal⁶⁵ amplitude superpositions of all dimer states within a topological sector. Here, the term topological sector is used in a restricted sense, where two dimer states belong to the same sector if they can be transformed into one another via resonance moves. On the kagome lattice, however, there seems²⁴ to be no distinction between topological sectors in this kinetic sense and the four topological sectors on the torus defined in terms of transition graphs or winding numbers (for recent reviews, the reader is again referred to Refs. 17 and

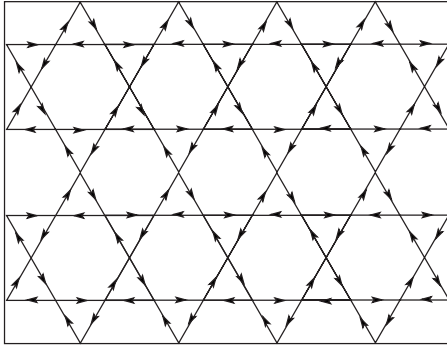


FIG. 7. Orientation of links on the kagome lattice used to fix the sign of valence-bond states. All links are oriented counterclockwise around the hexagon they belong to.

18). The QDM on the kagome lattice thus has exactly four degenerate ground states at the RK point. These were shown^{24,66} to be quantum liquids and argued to be in the \mathbb{Z}_2 universality class.

**2. Correspondence between dimer and valence-bond states:
Sign convention**

It is easy to elevate the dimer liquid states just described to states of superpositions of the spin- $\frac{1}{2}$ valence-bond states associated with each dimer basis state. As opposed to the preceding sections, the overall phases of valence-bond states now matter. A convention for the overall phase of a valence-bond state can be given by choosing an orientation for each link of the lattice. This then fixes the sign of each valence-bond singlet $[ij] = -[ji]$ on the link (i, j) , and thus of the valence-bond state, which is a product of singlets. A suitable way to orient links is to do so counterclockwise around each hexagon, Fig. 7. A resonance move can now be viewed as a cyclic permutation of spins long a loop of dimers. With the chosen orientation of links, the sign associated with the state is preserved by such moves. To see this, note that for any given 12-site cell of the lattice, flipping the orientation of the links touching boundary sites does not change the sign of any valence-bond state. This is so since any of the possible dimer coverings of the 12-site cell shown in Fig. 6 covers an even number of such links. But with this new orientation, Fig. 8, all links of the 12-site cell are oriented counterclockwise around the central hexagon. It is then clear that a cyclic permutation of spins around a dimer loop preserves the overall sign associated with the state. These observations motivate that the orientation chosen in Fig. 7 is natural in the following sense: If the basis of valence-bond states is defined using the sign convention derived from this link orientation, resonance moves on dimer states will translate into cyclic permutations of spins in the associated valence-bond states. One might now be tempted to write down an operator that performs such resonance moves when acting on valence-bond states. This is possible because of the linear independence of the valence-bond states. Furthermore, due to the general results of Sec. III, it is even possible to write such an operator as a sum of local terms (acting, say, on 19-site cells). Such an operator cannot serve as a physical spin- $\frac{1}{2}$

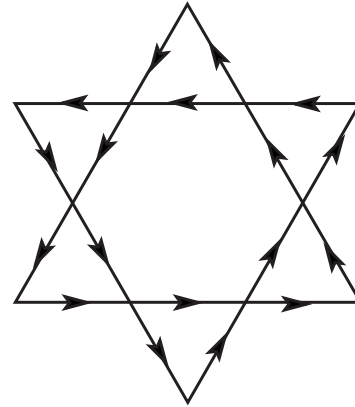


FIG. 8. Preferred link orientation for the 12-site cell. All links are oriented counterclockwise around the central hexagon. This orientation is consistent with the one shown in Fig. 7 for the entire lattice, as explained in the text.

Hamiltonian, however, since it would not be Hermitian with respect to the standard scalar product of the spin- $\frac{1}{2}$ Hilbert space. On the other hand, these observations do not rule out the possibility that one can construct a local (Hermitian) Hamiltonian whose exact ground states are just the equal amplitude superpositions of valence-bond states, within each topological sector and with the sign convention given here. This question will be addressed in the following.

3. Construction of the Hamiltonian

I now consider a Hamiltonian of the following form

$$H_{\text{RVB}} = - \sum_{\mathcal{C}} R_{\mathcal{C}}, \tag{17}$$

where the sum goes over all 12-site cells [as defined above, Fig. 2(a)] of the lattice and the $R_{\mathcal{C}}$ are certain Hermitian projection operators that enforce a “resonance” condition. For the time being, I will consider again a finite translationally invariant kagome lattice \mathcal{L} of toroidal topology. Hence the lattice \mathcal{L} has no boundary sites. The definition of the operators $R_{\mathcal{C}}$ is as follows: for a given 12-site cell \mathcal{C} , we first choose a sign convention for the valence-bond states $|D\rangle$ in accordance with the link orientation in Fig. 8. As explained the preceding section, this convention is consistent with the global sign convention chosen for valence-bond states on the lattice \mathcal{L} , Fig. 7. As discussed, every dimer pattern D on \mathcal{C} corresponds to one of two realizations of a certain loop around the central hexagon, Fig. 6. By D^* I now denote the other realization related to D by a resonance move. In analogy with the definition of the set $\mathcal{B}(\mathcal{C})$, Eq. (4), we can now define a set of “resonant states” $\mathcal{R}(\mathcal{C})$ via

$$\mathcal{R}(\mathcal{C}) = \{(|D\rangle + |D^*\rangle) \otimes |\psi_{D,j}\rangle; D \in \mathcal{D}(\mathcal{C}), j = 1, \dots, n_D\}. \tag{18}$$

Form the elements of $\mathcal{R}(\mathcal{C})$, we can linearly generate any state consisting of a resonant dimer loop with the free sites not touched by the loop in an arbitrary state. Note that $\text{free}(D) = \text{free}(D^*)$ and we may without loss of generality assume that $\psi_{D,j} = \psi_{D^*,j}$. This will avoid some unnecessary re-

dundancy under the exchange of D and D^* in Eq. (18).⁶⁷ In complete analogy with the space $\text{VB}(\mathcal{C})$ of valence-bond states on \mathcal{C} , one may now define the space $RL(\mathcal{C})$ of “resonance loop” states on \mathcal{C} , via

$$RL(\mathcal{C}) = \sum_{D \in \mathcal{D}(\mathcal{C})} \mathcal{H}(|D\rangle + |D^*\rangle). \quad (19)$$

The set $\mathcal{R}(\mathcal{C})$ linearly generates the space $RL(\mathcal{C})$, and in fact turns out to be a basis of $RL(\mathcal{C})$, although this will not be used in the following. It is natural to define the operator $R_{\mathcal{C}}$ to be the *orthogonal* projection onto the subspace $RL(\mathcal{C})$. Note that Hamiltonian (17) then has all the symmetries of the underlying lattice. In addition, it is invariant under $SU(2)$ rotations, since the space $RL(\mathcal{C})$ is $SU(2)$ invariant for each \mathcal{C} , for the same reasons stated below Eq. (3).

Let us now consider Rokhsar-Kivelson-type spin- $\frac{1}{2}$ wave functions defined as follows:

$$|\psi\rangle_{\Omega} = \sum_{D \in \Omega} |D\rangle, \quad (20)$$

where $\Omega \subset \mathcal{D}(\mathcal{C})$ contains all dimer coverings in a topological sector as described above. A similar type of state on the square lattice has been studied early on by Sutherland.⁵² In view of this, I will refer to the spin- $\frac{1}{2}$ wave function, Eq. (20), as the “SRK” state, to distinguish it from the RK state of hardcore dimers.

It is quite easy to see that Eq. (20) is indeed a ground state of Eq. (17) and, in fact, of every operator $-R_{\mathcal{C}}$. To see this, note that every state of the form

$$(|D_{\mathcal{C}}\rangle + |D_{\mathcal{C}}^*\rangle) \otimes |S\rangle, \quad (21)$$

is invariant under the action of $R_{\mathcal{C}}$, where again $D_{\mathcal{C}}$ is a dimer covering of the cell \mathcal{C} , and $|S\rangle$ is any state on $\mathcal{L} \setminus \text{supp}(D_{\mathcal{C}})$. The reason for this is entirely analogous to that given below Eq. (8). Equation (21) can be thought of as having a resonance loop on the cell \mathcal{C} . Since the eigenvalues of $R_{\mathcal{C}}$ are 0 and 1 by definition, Eq. (21) is thus a ground state of $-R_{\mathcal{C}}$. We now write Eq. (20) as a double sum

$$|\psi\rangle_{\Omega} = \sum_{D_{\mathcal{C}}} \sum_{\overline{D_{\mathcal{C}}}} |D_{\mathcal{C}}\rangle \otimes |\overline{D_{\mathcal{C}}}\rangle. \quad (22)$$

In here, the outer sum goes over all dimer coverings of \mathcal{C} , whereas the inner sum goes over all possible complements $\overline{D_{\mathcal{C}}}$ of $D_{\mathcal{C}}$ such that $D_{\mathcal{C}} \cup \overline{D_{\mathcal{C}}} =: D$ is a dimer covering of \mathcal{L} in the topological sector Ω . Note that for $D_{\mathcal{C}}^*$ instead of $D_{\mathcal{C}}$, the possible choices for $\overline{D_{\mathcal{C}}}$ are exactly the same, since $D_{\mathcal{C}}^*$ and $D_{\mathcal{C}}$ have the same support, and the dimer coverings $D_{\mathcal{C}} \cup \overline{D_{\mathcal{C}}}$ and $D_{\mathcal{C}}^* \cup \overline{D_{\mathcal{C}}}$ are in the same topological sector by definition, since they differ by a resonance move. We can thus also rewrite Eq. (22) as

$$|\psi\rangle_{\Omega} = \sum_{(D_{\mathcal{C}}, D_{\mathcal{C}}^*)} \sum_{\overline{D_{\mathcal{C}}}} (|D_{\mathcal{C}}\rangle + |D_{\mathcal{C}}^*\rangle) \otimes |\overline{D_{\mathcal{C}}}\rangle, \quad (23)$$

where the first sum now goes over (unordered) pairs $(D_{\mathcal{C}}, D_{\mathcal{C}}^*)$. Since Eq. (23) is a sum over states of the form Eq. (21), it is manifestly invariant under the action of $R_{\mathcal{C}}$. This proves that $|\psi\rangle_{\Omega}$ is a ground state of $-R_{\mathcal{C}}$ for each \mathcal{C} . Hence $|\psi\rangle_{\Omega}$ is a ground state of Hamiltonian (17).

4. Proof of uniqueness within the valence-bond basis

So far we have succeeded in constructing a Hamiltonian that has translational as well as $SU(2)$ invariance and has ground states of the SRK form, Eq. (20). At the same time, one may hope that this Hamiltonian lacks the extensive ground-state degeneracy of the generalized Klein models discussed in Sec. IV A. On the other hand, it is not yet clear how many ground states, Eq. (20), has, both within as well as outside the valence-bond subspace. In particular, since the construction of H_{RVB} closely follows that of the generalized Klein models, one may worry that some of the large degeneracy of the latter remains in the present case. In this section, it will be shown that this is not so, in the sense that at least *within* the valence-bond state space $\text{VB}(\mathcal{L})$ there is only one ground state of the form (20) within each topological sector Ω , on a finite kagome lattice \mathcal{L} . In Sec. V, it will be argued that either this already is the full degeneracy within the entire Hilbert space or one could find a perturbation such that this becomes the case. For simplicity, I will still assume periodic boundary conditions and comment on more general lattices in Sec. IV B 5.

Let us now consider a general state within the valence-bond subspace

$$|\psi\rangle = \sum_D a_D |D\rangle. \quad (24)$$

Suppose $|\psi\rangle$ is a ground state of H_{RVB} . I will now show that the assumption that $|\psi\rangle$ is *not* a linear combination of the states displayed in Eq. (20) then leads to a contradiction.

The fact that Eq. (24) is a ground state of H_{RVB} implies that it is a ground state of each individual operator $-R_{\mathcal{C}}$. For, the SRK states Eq. (20) have this property and would hence otherwise be lower in energy. This then implies that $|\psi\rangle$ is invariant under the action of each $R_{\mathcal{C}}$,

$$R_{\mathcal{C}}|\psi\rangle = |\psi\rangle. \quad (25)$$

On the other hand, if $|\psi\rangle$ is not a linear combination of states of the form (20), there must be a pair of dimer coverings D and D' related by a single resonance move, such that $a_D \neq a_{D'}$. For otherwise, it is easy to see that $|\psi\rangle$ would just be a superposition of SRK states, contrary to the assumption. So there is a 12-site cell \mathcal{C} such that the dimer loops contained in \mathcal{C} are $D_{\mathcal{C}}$ and $D_{\mathcal{C}}^*$ for D and D' , respectively, whereas the remaining dimers are denoted by $\overline{D_{\mathcal{C}}}$ and are the same for D and D' . We thus have

$$\begin{aligned} |D\rangle &= |D_{\mathcal{C}}\rangle \otimes |\overline{D_{\mathcal{C}}}\rangle, \\ |D'\rangle &= |D_{\mathcal{C}}^*\rangle \otimes |\overline{D_{\mathcal{C}}}\rangle. \end{aligned} \quad (26)$$

Suppose, now, that we can find a projection operator P which commutes with $R_{\mathcal{C}}$ and has the property that

$$P|\psi\rangle = a_D |D\rangle + a_{D'} |D'\rangle. \quad (27)$$

Then, Eq. (25) implies

$$R_{\mathcal{C}}(a_D |D\rangle + a_{D'} |D'\rangle) = a_D |D\rangle + a_{D'} |D'\rangle. \quad (28)$$

By definition of $R_{\mathcal{C}}$, we have

$$R_C(|D\rangle + |D'\rangle) = R_C(|D_C\rangle + |D_C^*\rangle) \otimes |\overline{D_C}\rangle = |D\rangle + |D'\rangle, \quad (29)$$

where again the fact was used that R_C leaves states of the form (21) invariant. Multiplying the last equation by $(a_D + a_{D'})/2$, subtracting from Eq. (28), and dividing by $(a_D - a_{D'})/2$ (which is nonzero by assumption) gives

$$R_C(|D\rangle - |D'\rangle) = |D\rangle - |D'\rangle. \quad (30)$$

By Eq. (26), the state on the right-hand side of the last equation is clearly of the general form

$$(|D_C\rangle - |D_C^*\rangle) \otimes |S\rangle, \quad (31)$$

with $|S\rangle$ some state on $\mathcal{L} \setminus \text{supp}(D_C)$ (here, $|S\rangle = |\overline{D_C}\rangle$). Equation (31) is the counterpart of Eq. (21) with the resonance loop replaced by an “antiresonance loop.” The latter are related to “vison excitations” in the QDM on the kagome lattice.²⁴ Two things now remain to be shown: (1) that an operator P with the desired properties can be found and (2) that no state of the form Eq. (31) can be invariant under the action of R_C , such that Eq. (30) leads to a contradiction. The second statement would easily follow if the set $\mathcal{B}(C)$ were linearly independent already for the 12-site cell. This is not so, and to avoid technicalities of this nature here, the proof is deferred to the Appendix. Here I will focus on the construction of the operator P , Eq. (27). Consider

$$P := \prod_{C'} P_{D_{C'}}, \quad (32)$$

where the product goes over all 19-site bricks C' that have no sites in common with the fixed 12-site cell C within which D differs from D' . The projection operators $P_{D_{C'}}$ are those defined in Eq. (6) and $D_{C'}$ is the restriction of D onto C' as always. It is then clear that R_C commutes with P since R_C only acts on C whereas P acts only on the complement \overline{C} of C in \mathcal{L} . Furthermore, it is clear that the action of P leaves the state $|D\rangle$ invariant by construction [cf. Eq. (10)], and the same is true for the state $|D'\rangle$, since D and D' do not differ on \overline{C} . It remains to show that P annihilates every valence-bond state other than $|D\rangle$ and $|D'\rangle$. To this end, it is best to introduce the arrow representation for dimer coverings,⁶⁰ as shown in Fig. 9. Here, one assigns an arrow to each lattice site, which points to the center of either of the adjacent triangles. These arrows are further subject to the constraint that each triangle must have either two inward pointing arrows or none. We associate a dimer with any link between two inward pointing arrows on any given triangle. It is easy to see that the allowed arrow states are in one-to-one correspondence with the dimer coverings of \mathcal{L} . Furthermore, the knowledge that a valence-bond state $|D''\rangle$ survives the action of the operator $P_{D_{C'}}$ already determines the arrows associated with D'' for all sites in C' : all dimers of D'' that are fully contained in C' must be identical to dimers in D , i.e., $D''_{C'} = D_{C'}$. Hence the arrows on sites touched by such dimers are determined while those of the remaining boundary sites of C' must point outward, i.e., away from C' . Note that the latter sites just make up the set $\text{free}(D_{C'}, C')$. Thus if $|D''\rangle$ survives

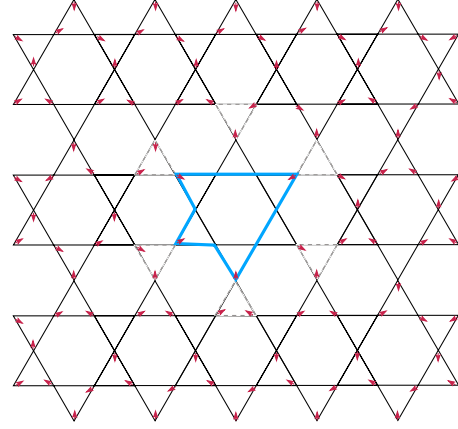


FIG. 9. (Color online) Arrow representation for the dimer covering shown in Fig. 1, except for the internal hexagon of the central 12-site cell C . The arrows shown are those that are uniquely determined by the condition that the associated valence-bond state survives the action of the operator P , Eq. (32). The dashed triangles indicate links that belong neither to the cell C nor to any 19-site brick C' that has no overlap with C . The arrows at the corners of the dashed triangles are nonetheless all determined (see text) and this determines the loop formed by the dimer configuration on C (fat line).

the action of P , the arrows corresponding to D'' are determined for all sites in \overline{C} . This follows since any such site belongs to a 19-site brick C' that has no overlap with C (cf. Fig. 9), provided that the lattice is sufficiently large (in both directions), which will be assumed in the following. Furthermore, as is apparent from Fig. 9, this also determines the arrows of D'' on boundary sites of C , since when two arrows of a triangle are determined, then so is the third. Hence, the fact that $|D''\rangle$ survives the action of P determines all the arrows associated with D'' except those on the interior sites of C . The arrows thus determined must be identical to those of D and D' since the corresponding valence-bond states $|D\rangle$ and $|D'\rangle$ likewise survive the action of P . However, as mentioned above, the arrows on the boundary sites of C determine the set $\text{free}(D''_{C'}, C)$ and thus determine the loop type associated with $D''_{C'}$. The remaining choices for the arrows on the internal sites of C then correspond to the two possible realizations of this loop, which then precisely lead to the dimer coverings D and D' . Hence, D'' must be equal to either D or D' . This concludes the proof of the two properties required of P , namely, $[R_C, P] = 0$ and Eq. (27). Note that the construction of the operator P is essentially possible due to the linear independence of the valence-bond states on \overline{C} , even though \overline{C} is not regular. (All sites of \overline{C} belong to a 19-site brick but not all links of \overline{C} . Cf. Fig. 9 and the discussion in Sec. III B.) I state the results of this section as

Theorem III. Let \mathcal{L} be a sufficiently large finite periodic kagome lattice. Then the SRK states, Eq. (20), and their linear combinations are the only ground states of the Hamiltonian H_{RVB} , Eq. (17), within the valence-bond subspace $\text{VB}(\mathcal{L})$.

5. Possible generalizations of Theorem III

The most important generalization of Theorem III one can consider relates to the obvious question whether or not the

uniqueness of the SRK ground states holds within the entire Hilbert space. Here I will briefly show that this question is naturally analyzed in two steps, the first of which is just the uniqueness within the valence-bond subspace, as stated in Theorem III. The second step, which is the extension of this statement to the full Hilbert space, may likely be analyzed in terms of the (arguably simpler) generalized Klein model, Eq. (15), for the 12-site cells, now denoted $H_{GK,12}$. More precisely, the following implication holds: if it can be shown that the generalized Klein model $H_{GK,12}$ has no ground states outside the NNVB subspace, then this is also true for H_{RVB} . This follows from the fact that every ground state of H_{RVB} is necessarily also a ground state of $H_{GK,12}$. To see this, it is best to focus on the local-density matrix ρ_C of a 12-site cell C for a ground state $|\psi\rangle$ of H_{RVB} . As discussed above, $|\psi\rangle$ is, in particular, a ground state of $-\mathcal{R}_C$. The ensemble described by ρ_C can thus have no weight outside the space $RL(C)$, Eq. (19). But since $RL(C) \subset VB(C)$, such a state is also a ground state of P_C , defined above Eq. (15). Since C was arbitrary, $|\psi\rangle$ is thus a ground state of $H_{GK,12}$. Needless to say, the reverse implication need not hold. Hence even if the generalized Klein model $H_{GK,12}$ has non-valence-bond ground states, this need not be true for H_{RVB} . Moreover, if *both* H_{RVB} and $H_{GK,12}$ had non-valence-bond ground states, it may be possible to find a generalized Klein model based on a larger cell C' (e.g., the 19-site bricks) which does not have such ground states. In this case, one can render the SRK ground states unique by adding this generalized Klein Hamiltonian (multiplied by an arbitrary positive constant) to H_{RVB} . However, it seems likely that $H_{GK,12}$ already has the desired properties in which case no such perturbation is needed. I note that for the usual Klein models, it seems that in all well-studied cases, the NNVB ground states are either unique,²⁶ or there are obvious (i.e., simple product state) exceptions, such as the one shown in Fig. 4. The latter does not seem to be the case for $H_{GK,12}$. One may thus be hopeful that the uniqueness of the NN valence-bond ground states can be explicitly demonstrated in this case, just as it has been possible for certain Klein models²⁶ (cf. also Ref. 68). A systematic study of this problem is reserved for future work.

Another natural generalization to consider is the application of the present construction to different lattices. For the square and honeycomb lattice, a different method is already available.⁵¹ However, the method discussed here is also applicable in these cases, see below. The solvable points of the underlying quantum dimer models on these bipartite lattices correspond to critical states. This may also be true of the corresponding spin- $\frac{1}{2}$ Hamiltonians. More akin to the kagome case discussed here is the triangular lattice, at least from the point of view of quantum dimer models.²³ In the triangular case, the RK point also describes a \mathbb{Z}_2 liquid phase, albeit at a first-order phase boundary. Furthermore, the RK points of quantum dimer models on bipartite lattices in three dimensions have been argued to describe stable critical phases.^{69–71} It seems desirable to generalize the methods developed here to all these cases.

On any lattice, the following generalization of the present construction suggests itself (cf. again Ref. 61). We consider a cell C which is sufficiently large. This implies that all types of resonance moves of the quantum dimer model on the

same lattice may take place within cells of this kind and also that the cell has nonvanishing interior. We consider SRK-type wave functions, and ask whether for these states, the resulting density matrix ρ_C of that cell is restricted to a certain subspace of $\mathcal{H}(C)$. The answer will in general be affirmative for large enough C . In fact, ρ_C will be restricted to (i.e., have no weight outside of) a certain subspace $RL(C) \subset VB(C)$. Here, $VB(C)$ is constructed just as before from all possible restrictions of dimer coverings to the cell C with free sites in an arbitrary state. $VB(C)$ may be used to define generalized Klein models, as discussed in Sec. III B. To construct $RL(C)$, we introduce equivalence classes on the set $\mathcal{D}(C)$ of dimer coverings of C , where two coverings belong to the same class if they are related by a series of resonance moves taking place *within the cell* C . In analogy with Eq. (18), the set $\mathcal{R}(C)$ of resonating states on C is then defined as equal amplitude superposition of valence-bond states within one equivalence class. Again, the state of the free sites is chosen from an arbitrary basis (but is fixed for any such superposition). It is then found that the space $RL(C)$ spanned by all states in $\mathcal{R}(C)$ contains all the nonzero weight of the local-density matrix ρ_C in an SRK-type state. These states are thus ground states of a Hamiltonian constructed in analogy with Eq. (17). It is clear that these observations are completely analogous to those made for the kagome lattice above, except that a language free of density matrices has been given preference there. Likewise, a discussion of the uniqueness of the SRK ground states on general lattices should in most cases be feasible along lines similar to those for the kagome case discussed above.

Finally, it is expected that the findings of the preceding sections are not limited to periodic lattices but can be carried over to reasonably benign lattices with an edge. On any such lattice, one would want to add a prescription for boundary terms of Hamiltonian (17). For, in the absence of translational symmetry, there is no reason why these should be identical to the bulk terms already defined. In fact, for 12-site cells C that lie at the boundary of the lattice, one would again want to alter the definition of the set $\mathcal{R}(C)$, Eq. (18), by discarding all dimer coverings $D \in \mathcal{D}(C)$ where some boundary sites of \mathcal{L} are not touched by a dimer. Otherwise, there would be gapless edge excitations, where spins at boundary sites are put into an arbitrary state and do not participate in valence bonds. Such gapless edge modes are not generically present in a \mathbb{Z}_2 topological state. However, the fact that the above modification easily gets rid of these modes indicates that a generic edge perturbation would do the same. A merely technical subtlety arises in the construction of the operator P , Eq. (32), in the proof of Theorem III. When a 12-site cell C is near the edge, it may not be possible to cover its complement \bar{C} by 19-site bricks. One may, however, find other “linear-independence bricks” near the edge, which may be smaller than the 19-site bricks, since for edge cells C' the set $\mathcal{B}(C')$ would be subject to the same truncation discussed above for the set $\mathcal{R}(C)$.

V. DISCUSSION OF THE PHYSICAL IMPLICATIONS OF THE SOLVABLE POINT

In the above, a point in the phase diagram of local $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonians on the kagome lattice

has been identified for which exact ground states can be found. A proof of the uniqueness of these ground states within a restricted Hilbert space has been given. It has been argued that this uniqueness likely holds within the full Hilbert space, possibly involving slight modifications of the Hamiltonian, and strategies have been outlined to prove this. Some interesting questions remain to be resolved by future work, which will be addressed only at a qualitative level here. These questions are: “is there a gap between the ground-state sector and the excited states?” “How do correlations behave at long distances?” “Do the ground states break any symmetry?” These questions are intimately related, both to each other and to the question of how much of the properties of the kagome lattice QDM survives in the present realization of the RK point through spin- $\frac{1}{2}$ degrees of freedom. One may certainly hope that correlations largely carry over from the quantum dimer case, where it has been shown that any correlations between local operators are short ranged.⁶⁶ The main difficulty in generalizing this proof to the present case is the nonorthogonality of the valence-bond states. It seems that this may not change the physics much, since the overlap between two random valence-bond states tends to be very small, and is only appreciable for similar valence-bond configurations. (See, however, Ref. 72 for a discussion of the impact of the choice of scalar product in a related but different problem.) The expected behavior is thus that the ultrashort-ranged correlations of the QDM become exponentially decaying for the SRK states, Eq. (20). If so, this would preclude the existence of broken symmetry and would be a strong argument in favor of a gap. One may caution that special Hamiltonians are known which are gapless despite short-ranged correlations.⁷³ However, this is not generically expected, and the knowledge that on any finite torus the ground state precisely has a fourfold topological degeneracy, combined with the absence of symmetry breaking, would be quite compelling evidence that the low-temperature phase of Eq. (17) is the same as that of the kagome lattice QDM. The latter is known to be a gapped Z_2 liquid.²⁴ Establishing the existence of an energy gap directly will likely require numerical efforts. On the other hand, it seems possible that the particularly benign properties of the kagome lattice, which have given rise to strong exact statements about dimer correlation functions for its RK states,⁶⁶ may allow insights into correlations in the present case as well. This would require one to tackle the issue of nonorthogonality, as discussed above, and will be left for future work.

I note that one attractive feature of the kagome lattice quantum dimer model is the exact knowledge of all eigenstates,²⁴ including spinons⁷⁴ and Ising vortex^{41,75} excitations, called “visons” in the recent literature.⁷⁶ This does not carry over to the present case, as the associated spin- $\frac{1}{2}$ wave functions would not be eigenstates of Hamiltonian (17) for any obvious reasons. They would, however, be natural variational candidates. Note that on the other hand, these excitations do not disperse for the solvable quantum dimer model, but should do so in the present case, which is certainly the generic behavior (cf., e.g., Ref. 77).

The solvable Hamiltonian constructed here seems somewhat unrealistic due to the presence of operators that act on

12 spins at a time. It is not immediately clear how dominant such terms are when the local operator R_C is expanded in two-spin and higher-order exchange terms. While explicitly carrying out such an expansion would be worthwhile, there is no reason to assume that nearest-neighbor two-body processes will dominate. Even so, there is much to be said in favor of the usefulness of an exactly solvable “reference point.” First, if the solvable point turns out to be gapped, as is expected in the present case, it must lie within the interior of a phase. This phase will survive at least small perturbations in the direction of more realistic Hamiltonians and there is a distinct possibility that one may make these perturbations large enough to reach a realistic regime without encountering a phase boundary. Second, while establishing the existence of a liquid phase numerically is exceedingly difficult due to size limitations, it may be somewhat less so to establish that two points belong to the same phase, especially if this phase has a robust gap along a line connecting these points. Hence, to establish the properties of a single reference point may be of considerable benefit, even if the reference point itself is unphysical. Third, one may expect that higher-order exchange terms, as certainly present in Eq. (17), can be of considerable importance on the insulating side near a metal-insulator transition, where a Hubbard-type expansion parameter t/U is not small. Arguments of this type have been made,⁷⁸ at least for four-spin ring exchange terms, in the concrete example of the triangular antiferromagnet κ -(ET)₂Cu₂(CN)₃.¹⁰

It is worth noting that the recently studied “herbertsmithite” kagome antiferromagnet^{13–15} may fit into a picture based on a gapless “Dirac” spin-liquid state.^{79–81} The gapless nature of this system is indicated by various experimental probes, such as low-temperature susceptibility and specific heat. Magnetic impurities may however play an important role in these low-temperature properties (see, e.g., Ref. 82 for a brief summary of results). The presence of gapless excitations is also supported by some recent numerical studies,⁸³ though only for the singlet sector. A physical picture for gapless singlet excitations has been offered in Ref. 84.

If the nature of the low-lying excitations of the herbertsmithite compound is gapless, the low-temperature phase of this system does not seem to be directly related to the solvable point described in this work. Nonetheless, the existence of a solvable point of this kind may open up the possibility that a topological spin-liquid state can, in principle, be realized in $SU(2)$ -invariant kagome antiferromagnets, if there is some mechanism that generates sufficiently high-order spin couplings. I note that a very similar conclusion has been reached before in Ref. 85 based on a projective symmetry group⁸⁶ analysis of Schwinger boson states.

VI. CONCLUSION

In this work, the linear independence of nearest-neighbor valence-bond states on the kagome lattice has been proven using a method that may allow generalization to other lattices as well. Furthermore, capitalizing on techniques used in the proof, a class of spin- $\frac{1}{2}$ model Hamiltonians has been

constructed whose ground states simultaneously minimize the energy of noncommuting local projection operators. One variant of these projection operators leads to the notion of “generalized Klein models.” Another variant of these operators, with a more restricted image, leads to an $SU(2)$ -invariant local Hamiltonian whose ground states within the nearest-neighbor valence-bond manifold are uniquely given by the four topologically degenerate “Sutherland-Kivelson-Rokhsar” states on toroidal kagome lattices. It is argued that these ground states describe a \mathbb{Z}_2 topological quantum liquid with unbroken translational and $SU(2)$ -rotational invariance. This is based on the close analogy to similar ground states of a quantum dimer model on the same lattice, where a notion of rotational invariance is lacking. Questions pertaining to the uniqueness of the SRK ground states within the full Hilbert space have been reduced to properties of the generalized Klein Hamiltonians, whose detailed study is left to future work. I am hopeful that follow up work on the remaining questions raised in this paper will establish the existence of the \mathbb{Z}_2 topological phase within the phase diagram of $SU(2)$ -invariant spin- $\frac{1}{2}$ Hamiltonians on the kagome lattice, and possibly other lattices, beyond reasonable doubt.

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APPENDIX: EXCLUSION OF VISON CONFIGURATIONS FROM THE GROUND-STATE MANIFOLD

In this appendix, a technical lemma is derived which completes the proof of the uniqueness of the SRK ground states within the NNVB subspace, Theorem III. The proof of this theorem was based in part on the following observation: Consider a 12-site cell \mathcal{C} of the topology shown in Fig. 2(a). On this cell we consider a state of the general form (31), which I restate here as

$$(|D\rangle - |D^*\rangle) \otimes |S\rangle. \tag{A1}$$

Here, D is a dimer covering of the 12-site cell, D^* is its counterpart related to D by a resonance move. $|S\rangle$ is some state on the remaining sites of the lattice, $\mathcal{L} \setminus \text{supp}(D)$. Then, the statement to be shown is that no state of the form (A1) may be invariant under the action of the operator $R_{\mathcal{C}}$ constructed in Sec. IV B 3. The first step is to recast this statement as a property of the 12-site cell alone, disentangled from the remaining lattice. To this end, one may observe that it is sufficient to show that no state of the following form,

$$(|D\rangle - |D^*\rangle) \otimes |\psi\rangle, \tag{A2}$$

may be a ground state of the operator $-R_{\mathcal{C}}$, where $|\psi\rangle$ is a state on $\text{free}(D, \mathcal{C})$. Equation (A2) is thus a state on \mathcal{C} . One way to see this is to observe that in the state Eq. (A1), the local-density matrix $\rho_{\mathcal{C}}$ for the 12-site cell can be written as a

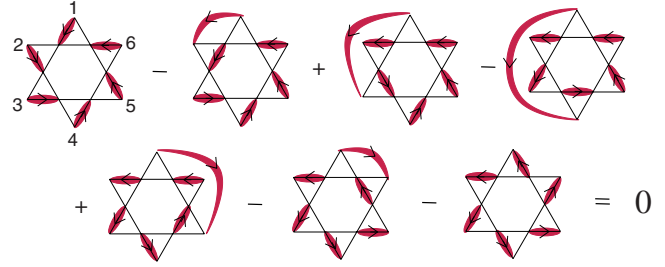


FIG. 10. (Color online) One of six linear dependences of the states in $\mathcal{B}(\mathcal{C})$ for the 12-site cell. The remaining five are related to the one shown here by rotation. See text and Eq. (A4).

sum of orthogonal projection operators onto states of the form (A2). It then follows that if no state of the form (A2) can be a ground state of $-R_{\mathcal{C}}$, then neither can any state of the form (A1). The latter is equivalent to the fact that no state of the form (A1) may be invariant under the action of $R_{\mathcal{C}}$.

All that remains to be shown is thus that no state of the form (A2) exists in the linear span of $\mathcal{R}(\mathcal{C})$, Eq. (18). Again, this would be trivial if the set $\mathcal{B}(\mathcal{C})$ consisted of linearly independent states but this is not the case for the 12-site cell considered here. Luckily, one finds that there are only six linear relations among the 730 states of the set $\mathcal{B}(\mathcal{C})$. These involve only the singlet sector and only five- and six-dimer loops. The two 6-dimer loop states are automatically singlets, whereas singlet five-dimer loop states have two free sites that form a singlet bond. Figure 10 graphically depicts one such linear relation, whereas the remaining five relations are obtained by rotating the diagrams in Fig. 10. In this figure, each bond denotes a singlet with orientations indicated by arrows. The validity of the identity shown in the figure follows easily from the following graphical identity for any pair of singlet bonds between four spins

$$\text{Two arcs} = \text{Two arcs} + \text{Two arcs} \tag{A3}$$

We may also write the six linear dependences in a more compact form as follows:

$$\begin{aligned} 6L - (1,2) + (1,3) - (1,4) + (1,5) - (1,6) - 6L^* &= 0, \\ 6L - (2,3) + (2,4) - (2,5) + (2,6) - (2,1) - 6L^* &= 0, \\ 6L - (3,4) + (3,5) - (3,6) + (3,1) - (3,2) - 6L^* &= 0, \\ 6L - (4,5) + (4,6) - (4,1) + (4,2) - (4,3) - 6L^* &= 0, \\ 6L - (5,6) + (5,1) - (5,2) + (5,3) - (5,4) - 6L^* &= 0, \\ 6L - (6,1) + (6,2) - (6,3) + (6,4) - (6,5) - 6L^* &= 0. \end{aligned} \tag{A4}$$

Here, $6L$ denotes the first six-dimer loop state in Fig. 10, $6L^*$ the last. A term (a, b) denotes a five-dimer loops state with free sites a and b joined by a singlet bond from a to b , where the boundary sites of \mathcal{C} are labeled as in the first graph of the figure. Here, b denotes the free site with a dimer on the same

triangle and a denotes the free site without a dimer on the same triangle. This determines all the remaining dimers, and nearest-neighbor valence bonds are by definition oriented counterclockwise around the central hexagon. The first line in Eq. (A4) is thus exactly the relation depicted in Fig. 10. Note that each (a, b) appears in one and only one line of Eq. (A4). We may further note

$$(a, b) = -(b, a)^*. \quad (\text{A5})$$

Here, $(a, b)^*$ denotes the state obtained from (a, b) by shifting all nearest-neighbor valence bonds along the loop they form but leaving the singlet between sites a and b untouched [hence the overall minus sign since the singlet bond between a and b has opposite orientations in (a, b) and (b, a)]. The fact that the relations in Eq. (A4) are the *only* linear relations between the states of $\mathcal{B}(\mathcal{C})$ can be shown analytically⁵⁴ by using Rumer-Pauling valence-bond diagrams.^{55–58} In addition, it is easy to verify this fact numerically, which I have carried out using Ref. 53. Let us now assume that a state of the form (A2) is contained in $RL(\mathcal{C})$, i.e., in the linear span of $\mathcal{R}(\mathcal{C})$. That is, we assume that there is a relation of the form

$$(|D\rangle - |D^*\rangle) \otimes |\psi\rangle = \sum'_{D',j} \lambda_{D',j} (|D'\rangle + |D'^*\rangle) \otimes |\psi_{D',j}\rangle \quad (\text{A6})$$

and want to show that this leads to a contradiction. Here, the prime restricts the sum to one of the two dimer coverings per loop. One may first observe that the dimer covering D on the left-hand side must correspond to a five- or six-dimer loop. Otherwise, there would be a nontrivial linear relation involving a three- or four-dimer loop valence-bond state, and such a relation does not exist, since Eq. (A4) is a complete set of linear relations. Furthermore, the state in Eq. (A6) cannot survive a projection onto the subspace of nonzero total spin

(which affects only the $|\psi\rangle$ factors). For otherwise, there would be a nontrivial linear relation involving nonsinglets, which again does not exist. Hence, Eq. (A6) must be of one of the following two forms,

$$6L - 6L^* = \lambda_6(6L + 6L^*) + \sum_{a' < b'} \lambda_{a',b'} [(a', b') - (b', a')], \quad (\text{A7a})$$

$$(a, b) + (b, a) = \lambda_6(6L + 6L^*) + \sum_{a' < b'} \lambda_{a',b'} [(a', b') - (b', a')], \quad (\text{A7b})$$

where Eq. (A5) was taken into account. Let us focus on Eq. (A7a) first. As a nontrivial linear relation, it must be possible to obtain Eq. (A7a) as a linear combination of the six relations in Eq. (A4),

$$\sum_{i=1}^6 \mu_i \ell_i, \quad (\text{A8})$$

where the ℓ_i represent the six lines of Eq. (A4). The requirement that (a', b') and (b', a') must enter with opposite signs results in the requirement that $\mu_i = -\mu_j$ for any $i \neq j$, which is evidently impossible. Next, let us try to obtain Eq. (A7b) from Eq. (A8). The situation is similar. There is now a single pair $(i, j) = (a, b)$ for which the relation $\mu_i = -\mu_j$ need not hold. For any $i \neq j$ with $(i, j) \neq (a, b) \neq (j, i)$, $\mu_i = -\mu_j$ still follows from the same reasoning as before. Still, this is impossible to satisfy. This concludes the proof that no state of the form (A2) can be expressed as a linear combination of the resonance loop states making up the set $\mathcal{R}(\mathcal{C})$, and hence no state of the form (A1) is invariant under the action of the operator \mathcal{R}_C , as explained initially.

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